The End of Classical Determinism

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Simple classical systems can be so unpredictable, both quantitatively and qualitatively, that they appear random. If there is a true source of randomness in such a system, the situation can be even more puzzling. This counterintuitive behavior, rich with temporal and geometric complexity, is still being uncovered, and has practical engineering consequences for systems as diverse as electronic oscillators and chemical reactors. This article develops a simple example system illustrating how classical systems can exhibit both quantitative and qualitative unpredictability, discusses the quantitative measures of such unpredictability, and places recent results developed at APL in context with the history of classical determinism.

INTRODUCTION

In 1820, the French mathematician Pierre-Simon de Laplace wrote in his *Analytical Theory of Probability:*

> An intellect which at a given instant knew all the forces acting in nature, and the position of all things of which the world consists—supposing the said intellect were vast enough to subject these data to analysis—would embrace in the same formula the motions of the greatest bodies in the universe and those of the slightest atoms; nothing would be uncertain for it, and the future, like the past, would be present to its eyes.

This is the doctrine of classical determinism at its most grandiose. Laplace, having worked for 26 years to successfully apply Newton's mechanics to the entire solar system, was making the ultimate extrapolation of physical law, as expressed in terms of differential equations: a set of such equations describing *any* system, together with a corresponding set of initial conditions, resulted in a complete prescription for determining the state of the system infinitely far into the future. More concretely, suppose a particular phsyical system at time t is described by a state vector $\mathbf{u}(t)$, and that the change in the state vector with time is given by some function $\mathbf{g}$ of the state vector and time; then there is an ordinary differential equation $d\mathbf{u}(t)/dt = \mathbf{g}(\mathbf{u}(t), t)$ describing the system. Given an initial condition $\mathbf{u}(t_0)$, the solution $\mathbf{u}(t)$ passing through $\mathbf{u}(t_0)$ at time t_0 reveals the past and future of the system merely by "plugging in" the desired value t.

Of course, Laplace fully realized that this picture represented an idealization, that there were practical limits imposed by computational capacity and the precision of the initial information. (He had to have realized this, having calculated *by hand* the predicted positions of many celestial bodies.) Nevertheless, for over two centuries something like this conceptual model was viewed as the triumph of a program begun by Newton for understanding the universe.

Any introductory physics student now knows that this picture is wrong. This century's ascendance of quantum mechanics, with its probabilistic interpretation of the wave function and its associated unaskable questions, doomed the deterministic picture of the universe. Much less well known is that the classical picture of determinism contained *within itself* a bankruptcy that rendered Laplace's picture useless, independent of any true randomness in nature. The limits imposed by computation and precision are much more serious than Laplace realized, even for extremely simple systems, and they destroy in short order the practical utility of the deterministic picture, even as an approximation. Because we still use classical descriptions to model macroscopic systems in many areas of science and engineering, the failure of classical determinism is much more than an interesting backwater in philosophy and pure mathematics.

Another French mathematician, Henri Poincaré, first revealed the problems with classical determinism around 1890, fully 10 years before the advent of quantum mechanics. The strangeness of Poincaré's findings[1] (now generally referred to as chaos) resulted in a relatively long delay before their generic importance was widely recognized. As late as 1981, a 750-page history[2] of the development of mathematical ideas contained a whole chapter on classical determinism, which was followed by a contrasting discussion of probability theory. Although Poincaré was discussed at length in the context of topology and celestial mechanics, he was not even mentioned relative to determinism.

Furthermore, the full ramifications of Poincaré's legacy are still being played out today, and some are quite spectacular. More delightful still (at least for those with a contrary nature), these most recent developments seem to have practical consequences in electrical engineering, chemistry, fluid mechanics, and probably other areas where classical approximations are still the most useful tools (even though a "correct" but impossibly complicated quantum description underlies the phenomena).

To cast the situation as a mystery, classical determinism was widely believed to have been murdered (maybe even tortured to death) by quantum mechanics. However, determinism was actually dead already, having been diagnosed with a terminal disease 10 years earlier by Poincaré. Having participated in a very late autopsy, I would like to describe some of the findings; the dangerous pathogens are still viable!*

I will illustrate how classical systems can be nondeterminstic for practical purposes using very simple one-particle systems like those analyzed in first-year undergraduate physics. Along the way, I will explain two quantifiable measures of classical nondeterminism, one that relates to *quantitative* predictability (how well you can predict, with specified precision, the exact system state), and one that relates to *qualitative* predictability (how well you can predict gross outcomes, where more than one outcome is possible). We will see how these measures indicate the failure of classical determinism, first in systems that have "only" chaos, and then in systems exhibiting the more serious (and more recently discovered) form of indeterminacy called riddled basins of attraction. Finally, I will discuss some practical areas where the recently discovered sorts of nondeterminism show up. (I hesitate to call these areas applications, because you don't really *want* to see this behavior in a practical system.)

* In reviewing this article, James Franson pointed out that classical dynamics may have its revenge on quantum mechanics anyway. Quantum chaos, an active field of research, is the study of the dynamics of classically chaotic systems as they are scaled down to sizes comparable to the de Broglie wavelength, where quantum effects become important. So far, such wave-mechanical systems have not been shown to exhibit chaos (which might not be surprising, given that the Schrödinger equation is linear). This creates a problem, because under the correspondence principle, quantum mechanics should be able to replicate all of classical mechanics in the limit as Planck's constant $\hbar$ goes to zero. Some physicists maintain that this discrepancy points to a fundamental flaw in the current formulation of quantum mechanics. Unfortunately, Joseph Ford, noted provocateur and one of the main proponents of this viewpoint, passed away in February 1995; it now seems less likely that this controversy will soon be resolved.

DEVELOPING AN EXAMPLE

The Simplest Case

Let's begin with a very simple system, a single unit-mass particle, and very simple physics, Newton's second law. (The system actually won't ever get much more complicated in this article, but the resulting behavior certainly will!) For a unit-mass particle moving in one dimension (coordinated by x), Newton's second law is

$$\sum_i F_i = \ddot{x}(t)\,, \tag{1}$$

or, the sum of the forces F_i gives the acceleration of the particle. (Newton's original "dot" notation denotes derivatives with respect to time.) Two forces act on the particle in this simplest example system: a small frictional force $-\gamma\dot{x}$ opposite (assuming $\gamma > 0$) and proportional to the particle velocity $\dot{x}$ and a force $-dV(x)/dx$ due to a scalar potential $V(x) = x^2$; this is a simple potential well, as shown in Fig. 1. (Incidentally, the formulation of forces in terms of scalar potential functions is another of Laplace's legacies to physics.) Thus, the second-order differential equation describing the motion of the particle in its potential energy well is

$$\ddot{x}(t) + \gamma\dot{x}(t) + 2x(t) = 0. \tag{2}$$

This linear equation of motion is so simple that it can be solved analytically, so the deterministic nature of the solution is manifest:

$$x(t) = \frac{1}{\omega}\exp\left(\frac{-\gamma t}{2}\right)\left\{\omega x(0)\cos\omega t + \left[\frac{\gamma x(0)}{2} + \dot{x}(0)\right]\sin\omega t\right\}, \tag{3}$$

where $\omega = \sqrt{8-\gamma^2}$.

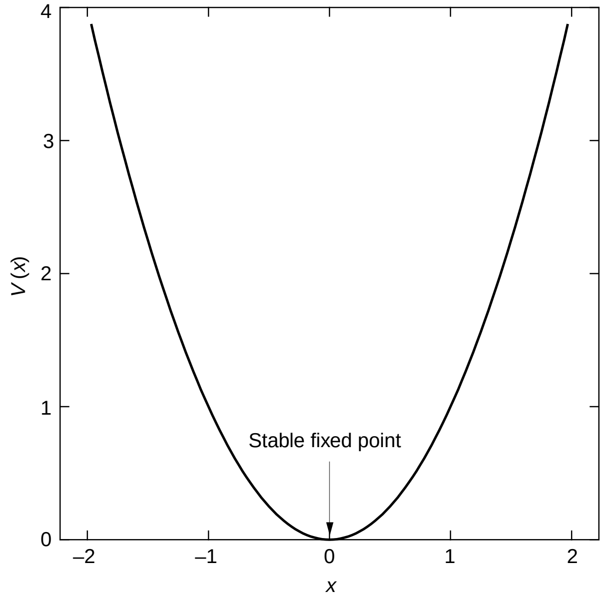


Figure 1. The simple quadratic potential for the single-particle dynamical system of Eq. 2. The system's attractor lies at the bottom of the well.

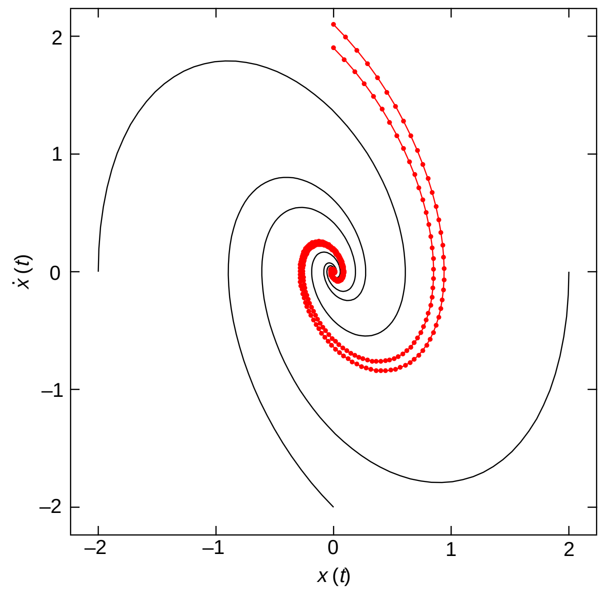


Figure 2. Typical phase-space trajectories of the dynamical system given by Eq. 2, which converge on the attractor at the origin. Nearby trajectories (red) stay nearby for all time.

One of Poincaré's innovations in analyzing dynamical systems was to consider the evolution of system trajectories in the space of possible initial conditions, what is now called phase space. In the case of Eq. 2, the phase space is coordinated by $x(t)$ and $\dot{x}(t)$, because one needs to specify both a starting position and a starting velocity to determine a particular instance of Eq. 3. Several sample trajectories, specific instances of Eq. 3 starting from different initial conditions, are shown in Fig. 2. Note that none of the trajectories cross. This nonintersection is true in general, and corresponds to the fact that the solution to Eq. 2 passing through any given initial condition is unique. Crossed trajectories would mean that the system was indeterminate: a given initial condition (the one at the point of intersection) could have two possible outcomes. Such indeterminacy is contrary to our classical physical intuition; nature "knows what to do" with any given setup. Further, note that the two trajectories shown in red start from nearby initial conditions. The dots along the trajectories show positions at successive, equally spaced instants in time. The neighboring trajectories "track" one another closely in both time and phase space.

The extent to which nearby trajectories track one another is one of the most important practical properties of a dynamical system. It governs how predictable the system is. There is a rigorous way to quantify this property, using quantities called Lyapunov exponents (see the boxed insert). Basically, the sign of the largest Lyapunov exponent describes whether neighboring trajectories converge or diverge. A negative exponent indicates convergence and good predictability. The convergence of trajectories in Fig. 2 is therefore consistent with the Lyapunov exponents for Eq. 2, which are both $-\gamma/2$.

Finally, note that all the trajectories of Eq. 2 end up at the origin of phase space, because the friction eventually dissipates any initial potential and kinetic energy, leaving the particle sitting at the bottom of the well. This may seem so obvious that using Lyapunov exponents to quantify predictability is drastic overkill, but this isn't true for later examples. This simple outcome does point to another important concept, however, that of an "attractor." An attractor is simply a set in phase space, invariant under the equations of motion, that is a large-time limit for a positive fraction of initial conditions in phase space. (There has actually been a lot of controversy over the definition of the apparently simple concept of an attractor, which is yet another indication of the subtleties uncovered by Poincaré. The definition given here is a paraphrase of one presciently proposed by mathematician John Milnor.[3]) One can see at once from Eq. 3 that $(x = 0, \dot{x} = 0)$ is invariant under Eq. 2; i.e., a system trajectory started at $(x = 0, \dot{x} = 0)$ and evolved forward under Eq. 2 stays put. Further, because everything in phase space ends up at the origin, the second condition is also satisfied, and the origin is indeed an attractor—specifically, a stable fixed point. Because the concept of qualitative predictability will also be of interest, we will now generalize our example system to include the possibility of distinct outcomes.

LYAPUNOV EXPONENTS

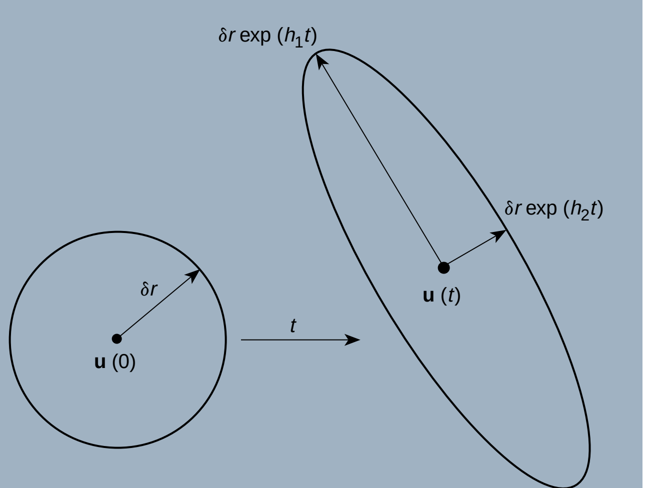
The relative stability of typical trajectories in a dynamical system is measured in terms of a spectrum of numbers called Lyapunov exponents, named after the Russian engineer Alexander M. Lyapunov. He was the first to consider that stability of solutions to differential equations could be a more subtle question than whether or not the trajectory tends to become infinite (as we usually mean for linear systems). A system has as many Lyapunov exponents as there are dimensions in its phase space.

Geometrically, the Lyapunov exponents for an m-dimensional system can be interpreted as follows: Given an initial infinitesimal m-sphere of radius δr, for very large time t, the image of the sphere under the equations of motion will be an m-ellipsoid (at some other absolute location in phase space) with semimajor axes of the order of $\delta r \exp(h_i t)$, $i = 1,2, \ldots ,m$, where the h_i are the Lyapunov exponents. This is illustrated for $m = 2$ in the accompanying drawing.

This set of numbers is a characteristic of the system as a whole, and is independent of the typical initial condition chosen as the center of the m-sphere. Clearly, if all of the Lyapunov exponents are less than zero, nearby initial conditions all converge on one another, and small errors in specifying initial conditions decrease in importance with time. On the other hand, if any of the Lyapunov exponents is positive, then infinitesimally nearby initial conditions diverge from one another exponentially fast; errors in initial conditions will *grow* with time. This condition, known as sensitive dependence on initial conditions, is one of the few universally agreed-upon conditions defining chaos.

Lyapunov exponents can be considered generalizations of the eigenvalues of steady-state and limit-cycle solutions to differential equations. The eigenvalues of a limit cycle characterize the rate at which nearby trajectories converge or diverge from the cycle. The Lyapunov exponents do the same thing, but for arbitrary trajectories, not just the special ones that are periodic.

Calculation of Lyapunov exponents involves (for nonlinear systems) numerical integration of the underlying differential equations of motion, together with their associated equations of variation. The equations of variation govern how the tangent bundle attached to a system trajectory evolves with time. For an m-dimensional system of ordinary differential equations, calculation of Lyapunov exponents requires the integration of an $(m^2 + 1)$-dimensional system (the m original equations, together with m additional equations of variation for each of m tangent vectors), together with occasional Gram–Schmidt orthonormalization for numerical conditioning.



Adding Another Outcome Introduces Nonlinearity

We will now work with a slightly more complicated system having a two-well potential $V(x) = (1 - x^2)^2$ (see Fig. 3). Again using Eq. 1 as a starting point, the equation of motion for this system is given by

$$\ddot{x}(t) + \gamma\dot{x}(t) + 4x(t)^3 - 4x(t) = 0 \,. \qquad (4)$$

The price of this generalization is considerable. We now have a nonlinear differential equation without an analytical solution, although extending the discussion of the simpler Eq. 2 gives us some information. We still have a two-dimensional phase space coordinated by x and $\dot{x}$, still have friction in the system, and have no source of energy except that carried by the initial conditions, so we can expect attractors at stable fixed points. (Less obviously, the Lyapunov exponents are still both $-\gamma/2$. Generalizing our two examples, one might expect that the sum of the Lyapunov exponents has some relation to the dissipative, or energy loss, properties of the system. This is, in fact, true in general.) In this case, though, there are *two* stable fixed points (and one unstable fixed point) at the critical points of the scalar potential. Thus, we can expect some initial conditions to reach their limit on the attractor at $(x = -1,\ \dot{x} = 0)$ while others end up at the other attractor at $(x = 1,\ \dot{x} = 0)$. Confining our attention for the moment to the section in phase space $\dot{x} = 0$, it is pretty easy to see how things will divide up. Any particle starting to the right of $x = 0$ and to the

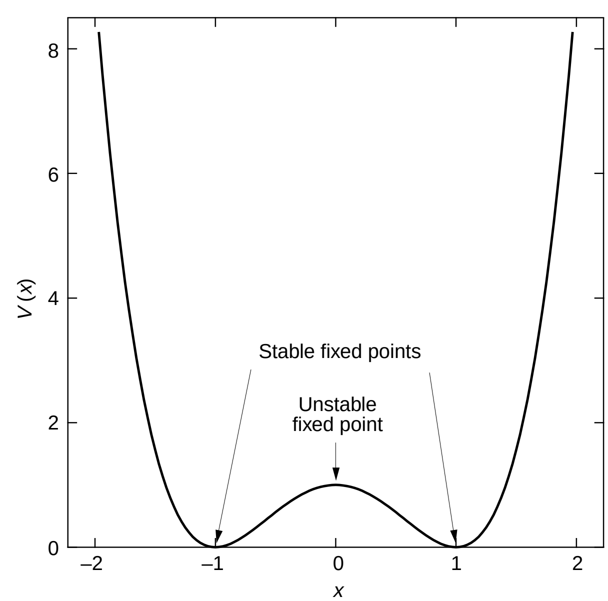


Figure 3. The more complicated potential for the single-particle dynamical system, Eq. 4, which has two attractors located at the stable fixed points.

left of some position depending on γ ends up falling to the right attractor. (For γ large enough, the system is drastically overdamped; that position is $x = \infty$.) There is then an interval still farther to the right where the particle has enough initial potential energy to make it over the central barrier once. Particles starting in this region will end up on the left attractor. An interval still farther to the right includes particles with enough energy to make it over the central barrier twice, ending up on the right attractor. So we envision an alternating set of starting intervals where the particle ends up on one of the two attractors. The situation to the left of the central barrier is symmetrical.

Allowing for nonzero initial velocity complicates things only a little more. Consider an initial condition just at the boundary between two of the intervals just discussed. That boundary represents an initial position where the particle has just enough potential energy to come to rest at the top of the central barrier, on the *unstable* fixed point at $x = 0$. Any nonzero velocity (i.e., higher energy) at the same starting location tends to push the particle over the top of the barrier and send it to the other attractor. Thus, the boundary between regions going to different attractors should be concave toward smaller initial positions. Figure 4 gives a map of the central region of phase space, color coded according to where the trajectory starting at a given initial condition ends up. A cut along $\dot{x} = 0$ shows the alternating intervals, together with the predicted curvature of the boundaries. Considered in the full phase space, the set of initial conditions going to a given attractor is called the "basin of attraction" for the specified attractor. The boundary between the two basins of attraction in Fig. 4 is itself a trajectory of Eq. 4, an atypical trajectory that ends on the unstable fixed point at the top of the barrier between the potential wells. Such an unstable equilibrium is frequently dismissed as physically irrelevant, as the probability of seeing it in a randomly initialized experiment is exactly zero. However, the whole behavior of the system in phase space is really organized around this atypical trajectory. This is another general property of dynamical systems.

To consider qualitative determinism, we must focus on the complexity of the boundary between basins of attraction. The basic question is: do two initial conditions started close together go to the same, or to different, attractors? We can quantify the answer to this question using a quantity called the uncertainty exponent (see the boxed insert). In this context, "uncertainty" refers to the fraction of pairs of randomly placed initial conditions that go to different attractors. Basically, the uncertainty exponent says how the uncertainty increases with the separation between the pairs of initial conditions. For the simple curvilinear boundary exhibited in the phase space of Eq. 4, the uncertainty exponent is 1. Thus the uncertainty goes up linearly with the separation between pairs of initial conditions. This is the answer we expect from our classical physical intuition: if you increase the accuracy of initial condition placement by a factor of 10, you expect a factor of 10 decrease in the uncertainty. But in general, it isn't necessarily so!

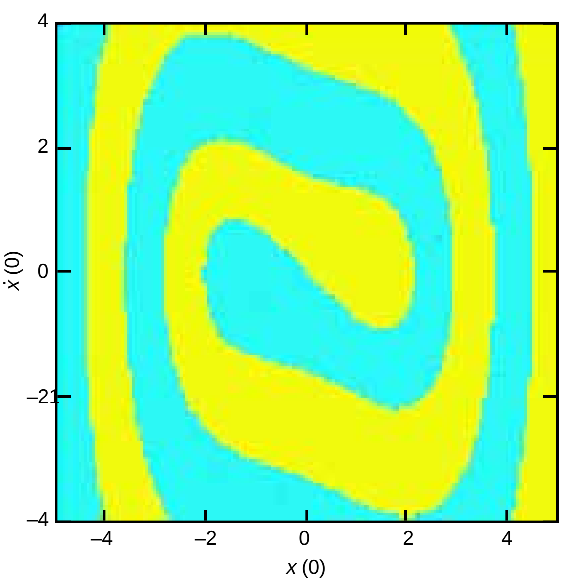


Figure 4. Basins of attraction in the phase space of Eq. 4. Initial conditions are color-coded according to their eventual destination. Initial conditions colored yellow tend to the attractor in the right well of the potential; those colored blue tend to the attractor in the left well. The attractors are indicated by small black crosses. There is a simple boundary between the basins of attraction.

Adding a Forcing Term Creates Chaos

We now further generalize our example system, adding a periodic forcing term of strength f and frequency ϕ to the sum in Eq. 1, so the equation of motion becomes

$$\ddot{x}(t) + \gamma\dot{x}(t) + 4x(t)^3 - 4x(t) = f \sin \phi t \, . \qquad (5)$$

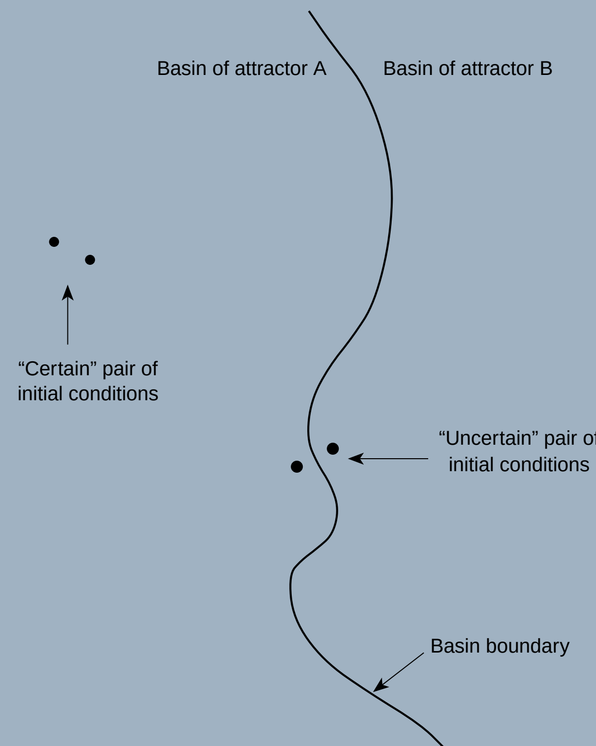
This is now a driven oscillator, similar to those studied in introductory differential equations courses, except, of course, that it is nonlinear, and so does not admit solutions that can be worked out with pencil and paper by students (or professors). If Eq. 5 were linear, we would expect the solution $x(t)$, after some initial transient, to be sinusoidal with the same frequency ϕ as the forcing term; only the amplitude would need to be calculated. First-order consideration of the nonlinearity might suggest additional complications, such as a periodic solution with more than one frequency component. Indeed, periodic solutions of all periods $T = 2m\pi/\phi$, $m = 1,2,3,\ldots$ are possible for Eq. 5, but we

UNCERTAINTY EXPONENTS

One of the problems in describing the complexity of a boundary between different basins of attraction is knowing when you are *on* the boundary. As discussed in the text, a point is on the boundary between basins if a system trajectory starting there never gets to either of the corresponding attractors. This is clearly *not* a constructive definition that would allow computation in a finite time.

An alternative approach is based on the idea that two system trajectories ending up at different attractors must have started on different sides of the boundary between basins. This procedure provides fuzzy information about the location of the boundary, but if repeated many times, it can provide good information about the complexity of the boundary. Imagine the following procedure. Two initial conditions, separated by a distance ϵ, are followed until they have arrived at definite outcomes. If each member of the pair goes to a different attractor, we call the pair uncertain. If many such pairs are randomly placed in a phase-space volume containing some piece of the boundary, a certain fraction $f(\epsilon)$ of pairs, depending on the value of ϵ, will be uncertain; we call $f(\epsilon)$ the uncertainty. (Since a boundary occupies zero phase-space volume, there is zero probability that a randomly placed initial condition will fall on the boundary). The uncertainty should decrease with ϵ; this is clearly what one expects intuitively. In fact, $f(\epsilon)$ (or at least its envelope) is proportional to ϵ^{α}; α is called the uncertainty exponent. For simple boundaries of the sort that we are used to drawing, $\alpha = 1$, as in the accompanying drawing. If the boundary is very complicated, it is possible for α to have values less than one.

The preceding description, while not one that most of us are familiar with, has the advantage that it *is* constructive. Pairs of initial conditions can be randomly chosen and numerically evolved forward using a computer. Estimating a value of $f(\epsilon)$ for a set of such ensembles at a decreasing sequence of ϵ values allows α to be determined to a specified statistical confidence. This approach was introduced by McDonald et al.[4] The uncertainty exponent has several desirable properties. First, it describes a property of the system with practical implications: the reliability with which a given outcome can be guaranteed, given a specified precision in placing initial conditions. Second, it allows determination of the fractal dimension of the basin boundary, since for a phase space of dimension D, and a basin boundary of dimension d, $D = d + \alpha$.



will focus on an even more fascinating type of behavior, one with no periodicity at all: *chaos*. For a positive-measure set of the parameters γ, ϕ, and f, the power spectral density of $x(t)$ will be broadband. A time series of $x(t)$ (see Fig. 5) looks like a sinusoid at the driving frequency f, with random modulation.

We now face a complication in representing the results in phase space. If one just displays the solution to Eq. 5 in the $\dot{x}$ vs. x plane as before, the curve so generated would cross itself. As discussed earlier, such intersections are impossible in phase space, so the phase space must have a higher dimension than two. The complication results from the explicit time dependence in Eq. 5. The phase space formalism requires us to describe systems as autonomous sets of ordinary differential equations. One can do that with a cheap trick, which immediately shows us the phase space needed. Equation 5 can be equivalently rewritten in canonical form as

$$\begin{cases} \dot{z}_1 = z_2 \,, \\ \dot{z}_2 = -\gamma z_2(t) - 4z_1^3(t) + 4z_1(t) + f \sin z_3(t)\,, \\ \dot{z}_3 = \phi \,. \end{cases} \qquad (5')$$

Here, we see explicitly that there are three coupled, autonomous, first-order ordinary differential equations, which require three initial conditions to specify the future evolution of the system.

Visualizing a three-dimensional phase space filled with continuous curves is difficult, advances in computer graphics notwithstanding. To avoid that problem, we can use a technique developed by Poincaré himself, now known as the Poincaré surface of section. Because the last equation in Eq. 5′ just indicates the monotonous increase in phase of the periodic forcing term, we can examine the solutions to Eq. 5′ only where they pass through a surface of section z_3 = constant. In that

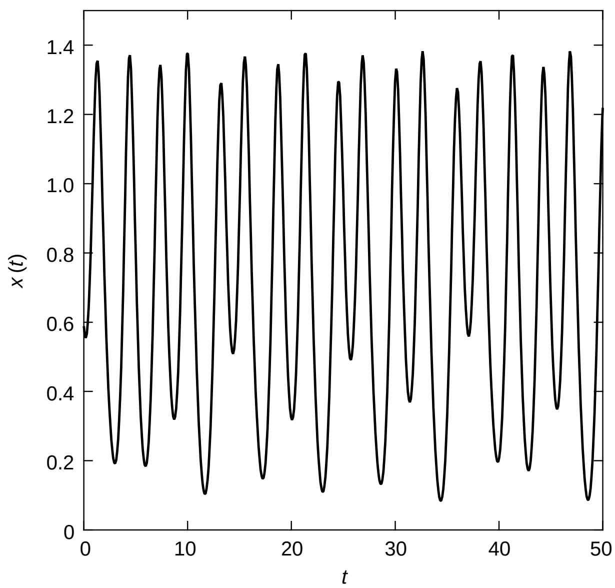


Figure 5. Time series of the state $x(t)$ from the periodically forced dynamical system of Eq. 5. Although the underlying forcing frequency is apparent, the time series is aperiodic because Eq. 5 is chaotic for the parameters specified in the text.

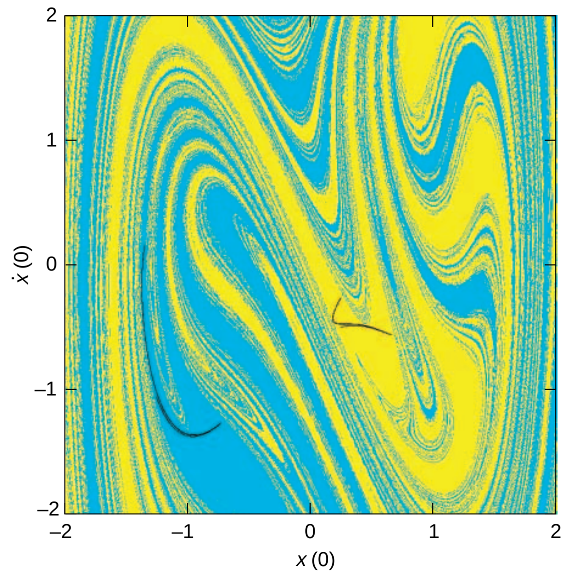


Figure 6. Basins of attraction in the phase space of Eq. 5. Initial conditions colored yellow tend to the attractor in the right well of the potential; those colored blue tend to the attractor in the left well. The strange attractors are shown in black. The boundary between the basins of attraction is fractal.

two-dimensional section, Eq. 5′ reduces to a *discrete* mapping, taking a point $(x,\ \dot{x})$ into another such point one forcing period later. There is a one-to-one correspondence between points in the surface of section and one-forcing-period-long pieces of continuous-time trajectory of Eq. 5′, so we lose no information by looking at this lower-dimensional representation of phase space. This mapping has an intuitive disadvantage, however: since it is a discrete mapping, trajectories of the mapping hop from place to (distant) place in the Poincaré section, and we have to abandon our picture of smooth curves parameterized by time such as those in Fig. 2.

Figure 6 shows the two possible qualitatively distinct outcomes of Eq. 5 for the parameter values $\gamma = 0.632$, $f = 1.0688$, and $\phi = 2.2136$, again corresponding to attractors confined to each of the two potential wells. Now, however, the attractors are not stable fixed points. The forcing, in a dynamic balance with the friction, allows for "steady-state" trajectories that are perpetually in aperiodic motion. The infinite number of intersections such trajectories have with the Poincaré section form a set with extremely high geometric complexity, called a fractal. A fractal (see the boxed insert) is a set with (typically) a noninteger dimension that exhibits a high degree of self-similarity, or scale invariance. An attractor having this strange type of geometry is called a strange attractor. The attractors shown in Fig. 6 do not look very complicated geometrically, but that is an artifact of the finite resolution of the computer-generated picture. In fact, both attractors have a fractal dimension $d_1 \approx 1.18$. (A symmetry in the system of Eq. 5 guarantees that the attractors will have the same dimension: $x \to -x, \dot{x} \to -\dot{x}$, and $t \to t + \pi/\phi$.) This dimension is smaller than that of an area-filling set ($d = 2$); in this respect, the attractors are like the stable fixed points of our earlier, less general examples. A general feature of systems with friction is that the long-time-limit motion of the system will occur on a set of zero phase-space volume.

The system's spectrum of the Lyapunov exponents has changed now, too. Motion on either of the attractors is now characterized by Lyapunov exponents of $h_1 = 0.135$ and $h_2 = -0.767$. The largest Lyapunov exponent is now positive, indicating that nearby initial conditions will diverge from one another exponentially fast (although only for a while; since the attractors are finite in extent, the exponential divergence must saturate eventually). Thus, we are now dealing with a chaotic system. Because of the positive Lyapunov exponent, we now know that our ability to predict the detailed evolution of the system is very limited. To be concrete, let's make a back-of-the-envelope calculation of the time over which we can compute the system trajectory with confidence. Suppose that we can compute with eight decimal digits of precision. Since the x coordinate for motion on the attractor is of order one, we can specify initial conditions only to an accuracy of 10^{-8}. An error of that magnitude will grow under the influence of h_1 to order one by $t = 136$, or only about 50 characteristic oscillations of the system. Thus, after about 50 oscillations, our original fairly precise

FRACTALS

We normally don't think of dimensions as something that one needs to *calculate*; we learn the dimensions of a few paradigmatic sets in grade school, and that's it! In fact, dimension is something that can be calculated, and there is a reason to do so. The most constructive approach to calculating the dimension of a set is a procedure called box-counting. A grid of boxes of side-length ϵ is overlaid on the set in question and the number $N(\epsilon)$ of boxes containing some of the set is counted. The procedure is repeated with a series of decreasing values of ϵ. The number N should increase with ϵ; in fact, $N(\epsilon)$ (or at least its envelope) is proportional to ϵ^{-d}, where d is the box-counting, or fractal, dimension of the set. Simple calculations confirm that this definition is consistent with the paradigms we memorized as children. For example, consider a line segment of length l. Then $N(\epsilon) \approx l/\epsilon$, so $d = 1$, as expected. Similarly, for a disk of radius R: $N(\epsilon) \approx \pi R^2/\epsilon^2$, so $d = 2$.

For such simple Euclidean sets as line segments and disks, this definition of dimension surely represents overkill. However, there are much more geometrically complex objects than those we learned about in Euclidean geometry. Consider the construction of the set C as shown in the drawing. We start with the unit interval, coordinated by the variable x. At the first step in the construction, we remove an open set consisting of all points with x coordinates in the open interval $(1/3, 2/3)$. At the second step, we remove all the points whose x coordinates are in either of the open intervals $(1/9, 2/9)$ or $(7/9, 8/9)$. At each successive step, the middle third of any remaining interval is removed, ad infinitum. After this procedure, C is not empty, since at least the endpoints of all the open intervals we removed are left behind. Thus, C is a set composed of an infinite number of points. We can make a table of $N(\epsilon)$ vs. ϵ for the box-counting procedure:

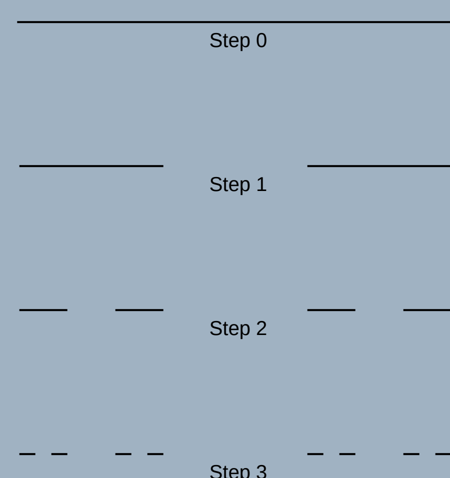
ϵ	$N(\epsilon)$
1	1
$1/3$	2
$1/9$	4
:	:
$(1/3)^n$	2^n
:	:

Applying our assumed scaling for the box count N, we see that $d = \log 2/\log 3 \approx 0.631$, which is clearly not an integer. Thus, a geometrically complex collection of points can have a fractal dimension strictly larger than the topological dimension (which is zero for a collection of disconnected points) but strictly smaller than the dimension of the embedding space (which is one for a line). Such a set is called a fractal. Fractals have been treated previously in the *Technical Digest*.[5]

Fractals also have the property of self-similarity. Appropriately rescaled parts of the set C look like complete copies of the original (for example, the part of C with x coordinates in $[0, 1/3]$, after being blown up by a factor of 3, is a complete copy of the whole set C. Many natural objects exhibiting some degree of scale-invariance (like coastlines or vascular networks) are well approximated by stochastic versions of fractals. Fractals are also frequently encountered in the context of nonlinear dynamical systems.

This last context is where one is most frequently confronted with the apparently self-contradictory nature of fractals. For an example of the conflict mentioned in the text between metric and topological measures, again consider the set C. It is easy to show that C has zero Lebesgue measure (for our purposes here, arc length); in constructing C, we *removed* a total arc length of $1/3 + 2(1/9) + 4(1/27) + \ldots = 1$. On the other hand, any point in C can be identified by a semi-infinite string of symbols, say b_n, where $n = 0,1,2,\ldots$ and $b_n \in \{R, L\}$, depending on whether the point in C is to the left (L) or right (R) of the interval removed at the n^{th} step in C's construction. Clearly, such semi-infinite strings can be put into one-to-one correspondence with the binary fractions in [0, 1] (just replace R by 0 and L by 1, and prepend the string with a binary point). Thus, in a metrical sense, the set C has "nothing" in it, although in a topological sense, C has as many elements as the unit interval!

Finally, it is worth mentioning that some of the fractals elsewhere in this article, namely the attractors of Eq. 5 and the boundaries of the basins of attraction of those attractors, are very similar to C crossed with a line; thus, their dimensions are between one and two.



knowledge of the initial condition has been reduced to the knowledge that the orbit is "somewhere on the attractor." Because the divergence in chaos is exponentially fast, increasing the precision of our computation does little good: if our computer keeps track of 20 decimal digits, we only get about 120 characteristic oscillations of predictability. This is certainly unsatisfying, but it is at least reassuring that the knowledge of the Lyapunov exponent allows setting quantitative limits on predictability. (The reader may wonder what the meaning of numerically calculated Lyapunov exponents, or indeed of any computer-derived statement such as that of Fig. 6, can be if the numerical stability of the system is so poor. This is a subtle question which did not have a very satisfactory answer until quite recently.[6] A detailed answer is beyond the scope of this article, but it turns out that for every computer trajectory, there is a real trajectory of the system starting from a slightly different initial condition. Thus statistical questions, at least, can be answered numerically, even if detailed, specific predictions are impossible. Finally, let me assure readers that accepting the pictures and numerical results in this article as accurate representations of the systems, at least to first order, will not mislead them.)

Note that, although he did not formulate his answer in terms of Lyapunov exponents, Poincaré essentially recognized this lack of long-term predictability in nonlinear systems at the end of the last century. (Actually, he was considering problems of orbital mechanics, which, unlike our example, are not dissipative and therefore do not admit attractors.) Even so simple a system as we are dealing with in this article could defeat the analytical intellect envisioned by Laplace, were its capacities anything less than infinite.

The latest change to our example system has also complicated the question of qualitative predictability. The division of phase space into basins of attraction is much more complicated than the simple boundary that we saw in Fig. 4. In fact, the boundary between the basins of attraction for the two attractors is a fractal also. Because a fractal is so complicated geometrically, it is hard to tell from a low-resolution picture on which side of the boundary an initial condition is placed if it is anywhere near the boundary. This difficulty is reflected in an uncertainty exponent α that is less than one for initial conditions near the boundary; in this case $\alpha = 0.85$. Thus, if one doubles the accuracy with which an initial condition is placed near the basin boundary, the uncertainty over a trajectory's destination attractor decreases by *less* than a factor of two. Fractal basin boundaries are another relatively recent discovery[4] that eroded the picture of classical determinism, this time as regards mere qualitative predictability. However, we can see from Fig. 6 that large areas of phase space contain no uncertainties over where one ends up. In particular, most places near the two attractors are "safe."

We can compare the division of phase space into basins of attraction with the familiar notion of continental drainage. Water poured on the ground east of the continental divide ends up in the Atlantic Ocean, whereas water spilled west of the divide flows to the Pacific. The details of the continental divide are complicated, so that if one is in the mountains of Montana, it might take considerable trouble to determine which direction water will flow, although in most places there is no uncertainty. It is even possible to imagine (say, in the absence of the Rocky Mountains) that in a few isolated places (like the coast of British Columbia), the continental divide could pass quite close to one of the possible attractors. Thus, water spilled near the Pacific Ocean could eventually flow to the Atlantic, although we would expect this to be an exceptional circumstance. The next generalization we make to our example system will shatter this picture completely; it will be as if the continental divide somehow expanded to come arbitrarily close to everywhere!

A Digression

This peculiar notion of a boundary being almost everywhere is so strange, especially in a physical context, that I will digress to explain how it arose. Poincaré discovered chaos and quantitative nondeterminism in the context of a well-defined question: is the solar system stable? How this question provided the roots of modern nonlinear dynamics is the theme of a fascinating popular book[7] that also details an almost unbelievable, and until 1993, successful conspiracy to suppress the fact that Poincaré's initial answer was incorrect!

Qualitative determinism fell as a result of not nearly so direct an assault. Mathematician James Alexander, while using James A. Yorke's nonlinear dynamics computer tool kit[8] DYNAMICS, noticed something unusual: a pair of attractors in a nonlinear system that appeared to cross. Remembering the uniqueness-motivated rule that trajectories in phase space cannot cross, it was hard to see how distinct attractors could intersect in this way. The basin of each attractor appeared to be riddled with "holes" on an arbitrarily fine scale; each hole was a piece of the other attractor's basin. Thus, the whole phase space appeared to be a boundary topologically. (In topology, a point is on the boundary between two sets if any neighborhood of the point contains points from each of the two sets.) Of course, one should be skeptical of such a startling hypothesis if it is based only on computer evidence, and Alexander could have been forgiven for dismissing the apparent result as an artifact of the computer. To Alexander and his collaborators' credit, however, they were able to prove that these riddled basins, as they called them at John Milnor's suggestion, really existed in the system.[9]

Meanwhile, Alexander was explaining his work in a series of talks at the Dynamics Pizza Lunch (more formally known as the Montroll Colloquium, a weekly seminar in nonlinear dynamics sponsored by the Institute for Physical Sciences and Technology at the University of Maryland, College Park, and attended by mathematicians, physicists, and engineers from the District of Columbia, Delaware, Maryland, and Virginia). In that informal setting, Alexander's work caused enormous controversy, much of it resulting from the specific mapping in which Alexander made his discovery. He had studied a nonanalytic mapping of the complex plane $z_{n+1} = z_n^2 - c\,\bar{z}_n$, where c is a complex-valued parameter, and the overbar denotes complex conjugation. In particular, serious concern arose over whether the lack of smoothness in the mapping had something to do with the results. If that were indeed the case, the peculiar results might well have no implications for classical mechanics, where the equations of motion are typically smooth.

Edward Ott (a physicist at the University of Maryland and previously my dissertation advisor) and I attempted to settle this controversy by finding riddled basins in a system of differential equations of the sort encountered in classical mechanics. We succeeded late in 1993.[10–13]

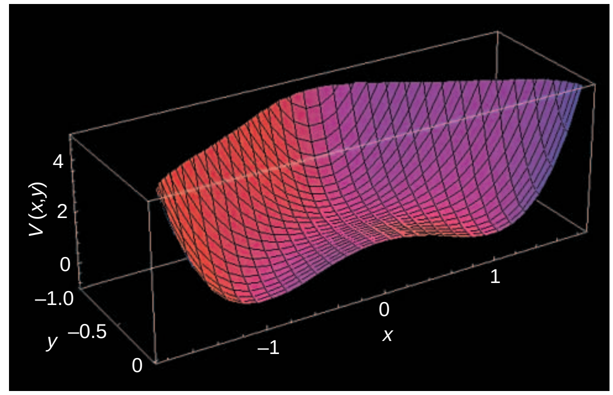


Figure 7. The potential $V(x, y)$ for the dynamical system given by Eq. 6a. The potential has reflection symmetry in y, so for clarity only the part in $y \le 0$ is shown. A section along $y = 0$ is the same as the potential in Fig. 3.

Increasing Dimensionality Allows Riddled Basins

A one degree-of-freedom system like Eq. 5, although it can exhibit such hallmarks of nonlinearity as chaos, is not very typical of physical systems. In particular, there are actual topological constraints on the behavior of dynamical systems in low-dimensional phase spaces, so we can't expect to see the whole range of pathology that nature has to offer in so simple a system. If only one more degree of freedom is added, the Pandora's box of riddled basins appears.

We will still work with a single particle moving under the influence of three forces: friction, an external periodic force, and a force due to a scalar potential. Now, however, we will allow the particle to move in *two* spatial dimensions, x and y. As before, Newton's second law is

$$\ddot{\mathbf{r}} = -\gamma\dot{\mathbf{r}} - \nabla V(\mathbf{r}) + \hat{\mathbf{x}}\, f \sin \phi t\,, \tag{6a}$$

where $\mathbf{r} = (x, y)$, $\hat{\mathbf{x}}$ is the unit vector in the x direction, and

$$V(x, y) = (1 - x^2)^2 + sy^2(x^2 - p) + ky^4\,. \tag{6b}$$

The potential V now has three adjustable parameters, s, p, and k, that control the behavior of the particle away from the x axis. The generalization to the scalar potential V, shown in Fig. 7, is not drastic: in the large-r limit, it is still a confining quartic energy well. In fact, the modified V has only one special feature that is essential for what we are about to find: it has a reflection symmetry in y. (The x-symmetry previously noted in the one degree-of-freedom system was not essential for any of the qualitative behavior discussed so far.) That symmetry implies a corresponding conservation law. If an initial condition starts on the phase-space surface $y = \dot{y} = 0$, it will remain there; therefore, the surface is an invariant surface for the dynamical system. Setting $y = \dot{y} = 0$ in Eqs. 6, we see that the equation of motion for the system is the same as Eq. 5, and will have the same attractors as Eq. 5, *for initial conditions in the invariant surface*. Considering the larger, four-dimensional Poincaré section coordinated by $x, \dot{x}, y,$ and $\dot{y}$, the question remains of whether the attractors of Eq. 5 are attractors in a *global* sense or not.

We can answer this question by considering the Lyapunov exponents for Eqs. 6 that result from an initial condition in the invariant surface. For the particular parameter choices $\gamma = 0.632$, $f = 1.0688$, $\phi = 2.2136$, $s = 20.0$, $p = 0.098$, and $k = 10.0$, the spectrum of Lyapunov exponents is, in order from largest to smallest, $h_1 = 0.135$, $h_2 = -0.012$, $h_3 = -0.644$, and $h_4 = -0.767$. Two of the Lyapunov exponents, h_1 and h_4, are the same as those we found for Eq. 5. They correspond to the convergence of nearby initial conditions separated *within* the invariant surface. As before, $h_1 > 0$ implies chaos. The other two Lyapunov exponents must therefore correspond to separations *transverse* to the invariant surface, i.e., one initial condition on the invariant surface and the other infinitesimally off of it. That these Lyapunov exponents are both negative, indicating convergence rather than divergence, means there are typical initial conditions off the invariant plane that are attracted toward it. Therefore, the attractors of Eq. 5 are attractors in the expanded phase space of Eqs. 6 as well. As it turns out, the potential of Eq. 6b admits no other attractors besides these two, so typical initial conditions in the phase space must eventually end up on one of the two attractors inherited from Eq. 5.

Now for the payoff! If we consider basins of attraction for these attractors, we find them hopelessly intermingled, as shown graphically in Fig. 8. Note, however, that this picture is different from earlier pictures,

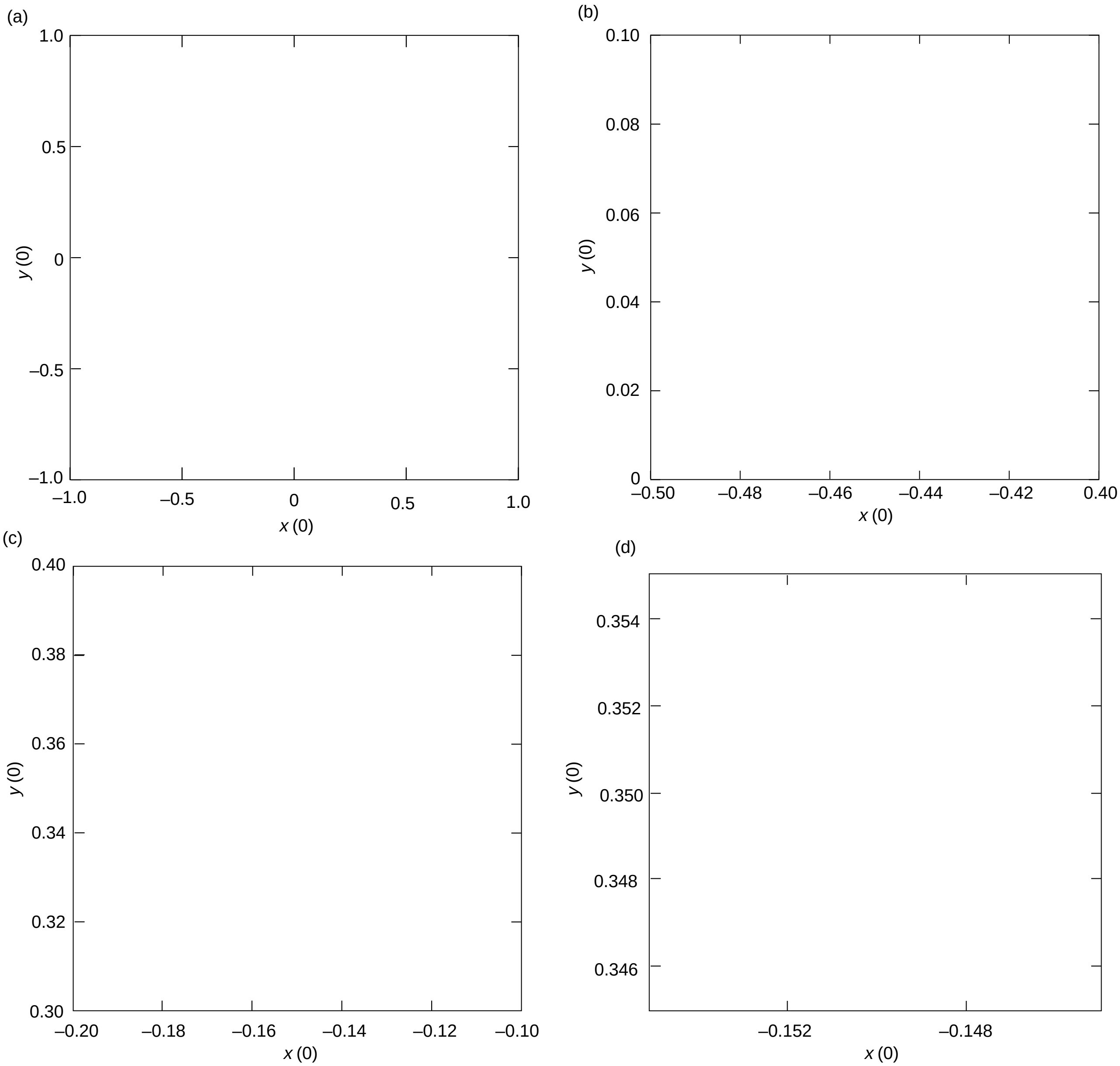


Figure 8. Basins of attraction in the phase space of Eqs. 6. Initial conditions colored yellow tend to the attractor in the right well of the potential; those colored blue tend to the attractor in the left well. The projections of the strange attractors are shown in black. (They are shown "end on" in this figure.) The two basins of attraction riddle each other everywhere. (b) and (c) show blowups of the red-bordered regions in (a), and (d) shows a blowup of the red-bordered region in (c), illustrating that the riddling persists to arbitrarily fine scales and is arbitrarily close to the attractors.

in that a two-dimensional picture cannot completely represent a four-dimensional phase space; the figure represents only a slice through the phase space. You will have to take my word that all sections look qualitatively the same (or write your own computer program to verify it). Each basin riddles the other on arbitrarily fine scales. If a given initial condition is destined for the attractor in $x > 0$, any phase-space ball surrounding it will have a positive fraction of its interior attracted to the other attractor, *no matter how small the ball is*. The uncertainty exponent for this system is around $10^{2\;2}$, so small that it is extremely difficult to measure precisely. The engineering implication of such a small uncertainty exponent is that if one could increase the reliability of initial condition placement by 10 orders of magnitude, the uncertainty in eventual outcomes would only go down by 25%! Before the discovery of riddled basins, I doubt that there was such a paradigm of engineering futility.

As just discussed, the entire phase space meets the topological definition of a boundary, yet at the same time, 100% of the volume of phase space is attracted to one of the two attractors. (The conflict between topological and metrical senses of "the whole thing" is one of the most confusing aspects of nonlinear dynamics. Although the set of initial conditions going to one of the two attractors has full Lebesque measure in the phase space, the set of initial conditions that never end up on one of the two attractors is dense in phase space, and uncountably infinite!) A particularly striking special case can be singled out: an initial condition can be arbitrarily close to one of the attractors and still end up going to the other attractor eventually. Returning to our continental drainage example, this switch to the other attractor corresponds to water spilled anywhere along the Atlantic coast having a positive probability of finding a path to flow all the way across the continent to the Pacific. This is not what we have come to expect from Newtonian mechanics.

RIDDLED BASINS OF ATTRACTION AND ON–OFF INTERMITTENCY

What are the observable implications of finding riddled basins in a physical system? Although the underlying equations of motion for a system like Eqs. 6 are deterministic, strictly speaking, the riddled geometry of its basin structure, coupled with unavoidable errors in initial conditions, renders it effectively nondeterministic, and in the worst possible way. Due to measurement errors, simple prediction of the qualitative outcome of even a perfectly modeled experiment would be impossible. Even the reproducibility of apparently identical trials (in reality, each starting at imperceptibly differing initial conditions) is problematical. Riddled basins can occur in many slightly different guises, such as a nonriddled basin riddling another basin, mutually riddled basins (as in the previous example; this case is called intermingled basins), and one or more nonriddled basins riddling one or more other basins, depending on the number of possible qualitatively distinct outcomes in the system. This plethora of possibilities can make recognizing what is going wrong with an apparently unreliable system very difficult.

The unavoidability of noise in practical applications makes the situation still more complicated. Returning to our example of Eqs. 6, imagine what would happen if a stochastic term were added to the equations of motion. An initial condition destined (in the deterministic system) for the attractor in $x > 0$ might appear to converge to the invariant surface's nonnegative side. Actually, the noise would keep the trajectory away from the attractor; but we already know that there is a positive-measure set of initial conditions arbitrarily near the invariant plane that lead to the other attractor. The noise-driven system must eventually hit one of these, and switch to the other attractor. The noisy picture thus includes an infinite sequence of arbitrarily long transients (very possibly longer than the practical waiting time in a physical experiment).

Actually, intermittent switching between attractors has been known in nonlinear systems for some time,[14] including the case where the intermittency is exclusively due to noise.[15,16] However, systems with riddled basins are closely related to systems capable of a particularly interesting type of intermittency, called on–off intermittency. To see this, we return to the deterministic Eqs. 6, at the very slightly different parameter value $p = 0.1$. At this parameter value, the Lyapunov exponent spectrum undergoes a qualitative change for trajectories on the invariant manifold: $h_1 = 0.135$, $h_2 = 0.004$, $h_3 = -0.636$, and $h_4 = -0.767$. Now there are two positive Lyapunov exponents. As before, we can see that the Lyapunov exponents corresponding to separations of initial conditions within the invariant plane are unchanged. However, one of the exponents corresponding to transverse separations is now positive, so the attractors of Eq. 5 can no longer be global attractors of Eqs. 6 in the whole phase space. The chaotic attractors of Eq. 5 have become chaotic repellers!

The typical behavior we expect from an initial condition is now the same as the noisy scenario discussed previously: a sequence of close approaches to the invariant plane, with potential switching between the regions of the repellers in $x < 0$ and $x > 0$. Typical time series from Eqs. 6 in this parameter regime are shown in Fig. 9. The time series for $y(t)$ makes the name "on–off intermittency" clear: the scalar y alternates between epochs of apparent quiescence (when the chaotic oscillations are so near the invariant plane as to be undetectable at the displayed resolution) and bursts into a larger region of phase space. This, too, is

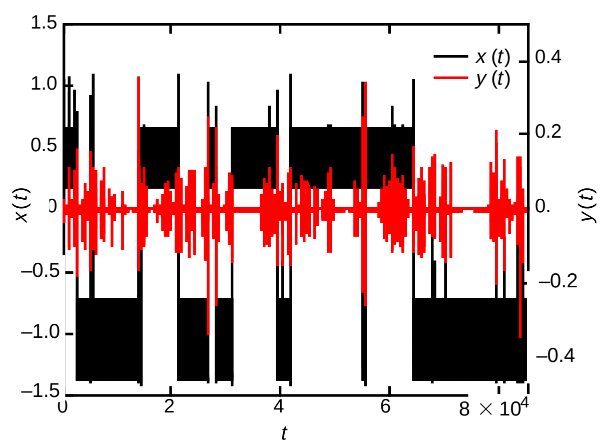


Figure 9. Time series for the states $x(t)$ and $y(t)$ of Eqs. 6 in a slightly different parameter regime than that for Fig. 8. Both $x(t)$ and $y(t)$ are strongly intermittent and chaotic. However, the time series $y(t)$ has long epochs where the oscillations are too small to observe (the "off" phase), separated by bursts of larger-scale activity (the "on" phase).

something we don't expect from Newtonian physics—a system that appears to be sitting there doing nothing for an arbitrarily long period of time that suddenly bursts into activity.

As it turns out, on–off intermittency has been independently rediscovered many times.[17–19] While listening to James F. Heagy, a Naval Research Laboratory colleague, give a Pizza Lunch seminar on the effects of noise on on–off intermittency,[20] Ott and I recognized that the analytical techniques he was using were very similar to those we were applying to the riddled basins problem. It soon became clear that riddled basins and on–off intermittency were different aspects of the same bifurcation in dynamical systems possessing an invariant surface (and therefore, a symmetry). We christened this situation a blowout bifurcation[21] because in either the hysteretic (riddled basins) or nonhysteretic (on–off intermittency) cases, the feature of interest is the possibility of orbits very near the invariant surface being "blown out" away from the surface to a different region of phase space.

Again, the multiple phenomena associated with riddled basins can occur in various combinations, depending on the underlying phase-space structure of the system in question, as well as on the relative strength of noise in the system, the precision with which parameters can be set and measurements made, and so on. Such a complicated situation makes analysis of these problems difficult, especially in an experiment where the entire system state may not be directly measurable. As a result, several collaborators in the area have worked out a number of diagnostics for these phenomena which can be applied in different experimental situations. We have worked out, for both noisy and deterministic cases, statistical measures of the basins of attraction, parametric scaling relations, geometric tests to be applied to time series, and scaling relations for power spectra of dynamical systems.[22–24] As a result, there is now a rather complete theoretical understanding of both riddled basins and on–off intermittency.

WHERE TO LOOK FOR RIDDLED BASINS

The key requirement for the occurrence of a blowout bifurcation (and therefore either riddled basins of attraction or on–off intermittency) is that the dynamical system must possess an invariant surface containing a chaotic set. To contain a chaotic set the surface must be at least two-dimensional for a Poincaré section, or three-dimensional for a continuous-time system (this is a consequence of the Poincaré–Bendixon theorem from the theory of ordinary differential equations). The overall dimension of the phase space must be at least one more than the invariant surface. Thus, a system of coupled, first-order differential equations must have at least four equations to exhibit these exotic dynamical behaviors (for example, Eqs. 6, written in canonical autonomous form, would have five first-order ordinary differential equations). This is not much of a restriction in the universe of classical systems.

The requirement for an invariant surface, however, is more restrictive. Typically, the system must possess some type of symmetry. (For example, Eqs. 6 have even symmetry in y.) Thus, riddled basins are by no means as ubiquitous a phenomenon in nonlinear systems as mere chaos (and a good thing, too, or the apparently nondeterministic behavior described previously would certainly have impeded the development of the scientific method). However, various symmetries are very common in physics and engineering, and the requirements for riddled basins are by no means so restrictive that they can be considered unnatural.

In fact, Ott and I predicted that the symmetry underlying the synchronization of coupled nonlinear oscillators would be a particularly fruitful area to explore for riddled basins.[21] Recently, many investigators have been interested in the behavior of synchronized, chaotic electronic oscillators as a basis for covert communications.[25] (Because chaotic systems generate broadband waveforms, they offer good opportunities to hide information. This is particularly appealing since it is fairly difficult to engineer high-power oscillators that behave in a linear fashion, anyway. Such systems have been under active development in several establishments.[26]) For example, consider two identical m-dimensional ($m > 2$) oscillators that are diffusively coupled:

$$\begin{cases} \dot{\mathbf{u}}_1 = \mathbf{F}(\mathbf{u}_1) + D(\mathbf{u}_2 - \mathbf{u}_1)\,, \\ \dot{\mathbf{u}}_2 = \mathbf{F}(\mathbf{u}_2) + D(\mathbf{u}_1 - \mathbf{u}_2)\,. \end{cases} \qquad (7)$$

The condition of synchronization ($\mathbf{u}_1 = \mathbf{u}_2$) represents an invariant surface for the system. The stability of that surface (i.e., whether a blowout bifurcation results in riddled basins or on–off intermittency) is determined by the details of the system, and by the coupling constant D. Riddled basins would further require multiple chaotic modes of oscillation for each oscillator. Either type of pathology would ruin the candidate system for communications purposes (since a departure from the invariant surface corresponds to a loss of the synchronization underlying the communications scheme).

Gratifyingly, our prediction was verified experimentally in both variations. Peter Ashwin, Jorge Buescu, and Ian Stewart at the University of Warwick confirmed that blowout bifurcations resulting in on–off intermittency could occur in coupled oscillators.[27] Heagy, Thomas Carrol, and Louis Pecora at the Naval

Research Laboratory experimentally demonstrated riddled basins of attraction in a similar circuit.[28] These experiments prove that the actual engineering of practical systems must account for seemingly esoteric pathologies of dynamical systems. Given that fundamental research is often questioned as to its relevance and applicability in practise, perhaps I can be forgiven for harboring a certain perverse delight in this setback for engineering colleagues.

In fact, experimental interest in riddled basins has exploded. Groups are working on this problem in the context of electronic oscillators, mechanical oscillators, simulation reliability, aerodynamic simulations, convective fluid systems, and reaction–diffusion systems. The last two areas might seem surprising, since until now I have discussed only finite-dimensional dynamical systems (and low-dimensional ones at that), whereas spatially distributed systems (described by partial, rather than ordinary differential equations) are intrinsically infinite-dimensional. However, they are rather natural generalizations of the diffusively coupled oscillators discussed previously (and we have predicted that riddled basins should be found in reaction–diffusion systems of a particular type[21]). This points to one of the most scientifically tantalizing aspects of the field of chaos: whether or not the complicated temporal behavior and geometric complexity so characteristic of low-dimensional nonlinear problems can contribute to the understanding of heretofore intractable problems of fluid mechanics, such as turbulence. That question is too broad and controversial to address in this article.

CONCLUSION

We have seen that extremely simple dynamical systems can behave in ways very much at odds with our intuition about the deterministic nature of classical physics, and furthermore that these exotic behaviors have some practical significance. It seems strange that so much could still be unknown about Newtonian mechanics more than 300 years after publication of the *Principia;* indeed, it took more than a century to answer the specific question posed to Poincaré about the solar system's stability.[29] This unexpected richness in classical dynamics provides at least one answer to the controversy surrounding the utility of the past 20 years' explosion of interest in chaos.[30–31] At a minimum, recent discoveries in nonlinear dynamics continue to expand the list of diagnostics for extreme intractability in practical systems.

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